

Electromagnetic Field and Cylindrical Compact Objects in Modified Gravity

Z. Yousaf^{*} and M. Zaeem-ul-Haq Bhatti[†]

*Department of Mathematics, University of the Punjab,
Quaid-i-Azam Campus, Lahore-54590, Pakistan.*

ABSTRACT

In this paper, we have investigated the role of different fluid parameters particularly electromagnetic field and $f(R)$ corrections on the evolution of cylindrical compact object. We have explored the modified field equations, kinematical quantities and dynamical equations. An expression for the mass function has been found in comparison with the Misner-Sharp formalism in modified gravity, after which different mass radius diagrams are drawn. The coupled dynamical transport equation have been formulated to discuss the role of thermoinertial effects on the inertial mass density of the cylindrical relativistic interior. Finally, we have presented a framework, according to which all possible solutions of the metric $f(R)$ -Maxwell field equations coupled with static fluid can be written through set of scalar functions. It is found that modified gravity induced by Lagrangians $f(R) = \alpha R^2$, $f(R) = \alpha R^2 - \beta R$ and $f(R) = \frac{\alpha R^2 - \beta R}{1 + \gamma R}$ are likely to host more massive cylindrical compact objects with smaller radii as compared to GR.

Key words: black hole physics– gravitation– hydrodynamics– methods: analytical.

1 INTRODUCTION

It seems established from direct observations (Perlmutter et al. 1999; Riess et al. 2007; Komatsu et al. 2011) of high red-shift supernovae as well as the cosmic microwave background fluctuations that the universe expands with acceleration at present state indicating the presence of dark energy. Thus, in the study of theoretical physics, the acceleration of the universe have presented some greatest issues. Such investigations provide motivation to discuss the modification in gravitational dynamics at cosmological scales. A number of attempts (Nojiri & Odintsov 2007b; Capozziello & Laurentis 2011; Nojiri & Odintsov 2011) have been made by many relativistic astrophysicists to describe the accelerating cosmologies at different epochs. Particularly, a mysterious type of energy dubbed as dark energy is involved in the explanation of the current universe acceleration. Still, there is no suitable solid theoretical outcomes for the source of this exotic distribution appearing at the current epoch.

In this paper, we have focused our attention at the direct addition of higher order curvature invariants in the Einstein-Hilbert (EH) action. The motivation to take such a study under consideration is that higher order curvature invariants emerge within strong gravity regions inside the

compact objects. Different authors have discussed the modification in the EH action with terms that are pretty effective in high-curvature regimes. Carroll et al. (2004) have investigated that cosmic acceleration can emerge with very small corrections of the form R^n in the usual gravitational action of general relativity (GR) with $n < 0$ and R as the curvature scalar. The modifications in gravity model describe the accelerated expansion but it also contains some instabilities (Chiba 2003). Dolgov & Kawasaki (2003) explored the instabilities in the matter due to extra curvature ingredients in the star configuration with a specific model of $f(R) = R - \frac{\mu^4}{R}$. Sotiriou & Faraoni (2010) have presented a comprehensive review of $f(R)$ theory of gravity describing its action, field equations, equivalence with other higher order theories, viability criteria and astrophysical applications including all its know formalisms namely, metric, Palatini and metric affine gravity.

The issue of gravitational collapse of massive stars has attracted many researchers in the study of gravitational physics as it is expected to play a vital role. Joshi, Dadhich & Maartens (2002) investigated that sufficiently strong shearing effects are responsible for the appearance of naked singularity (instead of black hole formation) during the collapse of a compact object. Milgrom (2009) studied that the isotropic and homogeneous models exist under certain conditions in the Palatini formalism of $f(R)$ theory during the stellar gravitational col-

^{*} zeeshan.math@pu.edu.pk (ZY)

[†] mzaem.math@pu.edu.pk (MZB)

lapse. Bamba, Nojiri & Odintsov (2011) explored the appearance of curvature singularity during the process of spherical collapse with $f(R)$ environment and deduced that viable $f(R)$ gravity models could become free of such singularities. Cembranos, de la Cruz-Dombriz & Núñez (2012) studied the collapsing phenomena within the framework of modified gravitational theories which represents an important tool for late-time cosmological acceleration. Ghosh & Maharaj (2012) found the non-static exact solutions for dust cloud in $f(R)$ gravity describing gravitational collapse. Sharif and his collaborators (Sharif & Bhatti 2015, 2014b,c; Sharif & Yousaf 2014c,a) have presented a effects of several fluid parameters in the modeling of gravitational implosion of relativistic interiors.

Most of the stars in the universe are main sequence stars which on evolutions collapses and leaves a compact object which is supported by pressure gradients in the star core. Recent observations show that many compact objects exist in nature like X-ray sources 4U 1728-34, X-ray burster 4U 1820-30, millisecond pulsar SAX J 1808.4-3658, X-ray pulsar Her X-1, PSR 0943+10 and RX J185635-3754, whose predicted radii and masses are not consistent with standard neutron star models. Some theoretical advances indicate that pressure at the interior of such stars are likely to be anisotropic as their density is greater than that of a neutron star, thus, radial and tangential pressures exist. Recently, different authors (Herrera & Santos 1997; Di Prisco et al. 2007; Bohmer & Harko 2013; Pinheiro & Chan 2013) have explored the models of compact stars with anisotropic fluid configurations. It is found that Fermi gas under the influence of magnetic field generates pressure anisotropy (Chaichian et al. 2000; Martinez, Rojas & Cuesta 2003). One can also examine the existence of anisotropy in wormholes (Morris & Thorne 1998) and in gravastars (Cattoen, Faber & Visser 2005; DeBenedictis et al. 2006) which are known as peculiar solutions of the field equations. Ryblewski & Florkowski (2011) have formulated a framework for strongly dissipating and highly anisotropic hydrodynamical system. They observed that the thermalization of the system during the evolution of matter slows down due to anisotropy by fixing initial profile of energy density and varying initial entropy density, simultaneously.

Electromagnetic studies in curved spacetimes have a long primordial history from the direct coupling of Einstein (or modified) and Maxwell fields which is interpreted as the scattering of electromagnetic waves due to spacetime curvature. Moffat (1979) derived the solutions of field equations in a new gravity for a static spherically symmetric star configuration, when the electromagnetic effects are included in the Lagrangian. Dehghani (2003) presented a class of solutions for the field equations in a modified gravity theory with negative cosmological constant and electromagnetic field which are interpreted as black brane solutions. Tsagas (2005) discussed the electromagnetic field for general curved spacetime filled with perfect fluid by using a covariant approach in GR. Herrera, Di Prisco & Ibáñez (2011b) explored a crucial role of electromagnetic field on the study of some scalar functions obtained from the splitting of the curvature tensor and named them as structure scalars. These structure scalars are endowed with different physical meanings in the study of relativistic astrophysics, particularly, under the influence of electromagnetic field (Sharif & Bhatti 2012).

Herrera & Barreto (2012) found that the electromagnetic radiations or electromagnetic waves does produce the rotation in the system and consequently is responsible for frame dragging. Herrera, Di Prisco & Ospino (2012) brought in the consequences of structure scalars for in the formulation of several fundamental properties of cylindrically symmetric relativistic interiors. Sharif & Yousaf (2015b) explored influence of $f(R)$ extra curvature terms on the cylindrical gravitational implosion with the help of $f(R)$ structure scalars.

Here, a systematic construction of the scalar functions have been presented for charged cylindrical relativistic object in the background of $f(R)$ gravity model. Particularly, we have explored the effects of fluid variables as well as $f(R)$ corrections on the structure and evolution of radiating cylindrical compact object. In this respect, set of governing equations are calculated and expressed in terms of $f(R)$ structure scalars in order to emphasize the importance of such scalars in the dynamical analysis of collapsing objects. We also highlighted their significance in the modeling of anisotropic and isotropic cylindrical collapsing system.

The paper is organized in the following fashion. After the formulations of basic equations for cylindrical anisotropic radiating system in section 2. Section 3 explore modified versions of structure scalars as well as conservation laws framed within viable $f(R)$ model. We then formulate two important evolution expressions that will make correspondence between Weyl scalar, dissipating and non-dissipating fluid parameters and some scalar variables. Section 4 is aimed to develop modified form of cylindrical mass function as well as dynamical transport equation. We also investigate the influence of inertial thermal effects on the mass density during evolution of stellar interior. In section 5, we discuss anisotropic as well as isotropic static cylindrical models. The results are summarized in the last section.

2 CHARGED DISSIPATIVE ANISOTROPIC RELATIVISTIC CYLINDERS

The notion of $f(R)$ gravity as a possible modifications in the gravitational framework of GR received much attention of researchers. This theory provides numerous interesting results in the field of physics and cosmology like plausible explanation to the accelerating cosmic expansion (Nojiri & Odintsov 2007b; Capozziello & Laurentis 2011; Nojiri & Odintsov 2011). The main theme of this theory is to substitute an algebraic general function of the Ricci scalar, $f(R)$, instead of the cosmological constant in the standard EH action. It can be written as

$$S_{f(R)} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S_M, \quad (1)$$

in which κ is the coupling constant, g is the determinant of the tensor, $g_{\alpha\beta}$, while S_M is the action for matter density. When the curvature has a sufficiently large value, then $f(R)$ in the above equation attains some constant value thus behaving as a cosmological constant (Λ). However, in other epochs, the non zero value of $f_R \equiv \frac{df}{dR}$ give rise to complicated dynamics, thereby presenting it as a phenomenology rich gravitational theory (Sotiriou & Faraoni 2010). One can obtain $f(R)$ field equations by varying Eq.(1) with

respect to $g_{\alpha\beta}$ and is given as follows

$$R_{\alpha\beta}f_R - \frac{1}{2}f(R)g_{\alpha\beta} + (g_{\alpha\beta}\square - \nabla_\alpha\nabla_\beta)f_R = \kappa T_{\alpha\beta}, \quad (2)$$

where $R_{\alpha\beta}$ and $T_{\alpha\beta}$ are Ricci tensor and standard energy momentum tensor, respectively. The above field equation is a 4th order differential equation due to the existence of second order derivatives of $f_R = f_R(R)$ (It is worthy to mention that f_R contains further second order derivatives of the metric functions). Some researchers have dubbed f_R as scalaron that indicates new scalar degree of freedom. Its equation of motion can be specified by taking the trace of Eq.(2) as

$$\square f_R = \frac{1}{3}(2f + R + \kappa T - Rf_R), \quad (3)$$

where $T \equiv T^\beta_\beta$. On setting, $f_R \rightarrow 0$, $f(R) \rightarrow \Lambda$, one can get a well-known result, i.e., $R = -(\kappa T + 2\Lambda)$. In terms of Einstein tensor, $G_{\alpha\beta}$, Eq.(2) can be written as

$$G_{\alpha\beta} = \frac{\kappa}{f_R} (T_{\alpha\beta}^{(D)} + T_{\alpha\beta}), \quad (4)$$

where

$$T_{\alpha\beta}^{(D)} = \frac{1}{\kappa} \left\{ \nabla_\alpha \nabla_\beta f_R - \square f_R g_{\alpha\beta} + (f - Rf_R) \frac{g_{\alpha\beta}}{2} \right\},$$

is an effective $f(R)$ gravitational interaction.

We consider a non-rotating diagonal cylindrical relativistic system characterized by the following line element (Herrera, Di Prisco & Ospino 2012)

$$ds^2 = -A^2(dt^2 - dr^2) + B^2dz^2 + C^2d\phi^2, \quad (5)$$

filled locally anisotropic matter configuration that is dissipating in the mode of heat radiations. Here A , B and C are the functions of t and r . The mathematical formula describing fluid distribution within the cylindrical relativistic celestial object is

$$T_{\alpha\beta} = (\mu + P)V_\alpha V_\beta + (P_\phi - P_r) \left(K_\alpha K_\beta - \frac{h_{\alpha\beta}}{3} \right) + (P_z - P_r) \times \left(S_\alpha S_\beta - \frac{h_{\alpha\beta}}{3} \right) + q_\alpha V_\beta + P g_{\alpha\beta} + q_\beta V_\alpha. \quad (6)$$

The evolution of this fluid distribution is characterized by timelike 3D boundary, Σ . Here μ is a matter energy density, P_r , P_z and P_ϕ are the corresponding principal stresses, while q_α is a four vector radiating heat along the radial direction. The unitary vectors, S_α , K_α , V_α and L_α , are configuring with the fact that they determine canonical orthonormal tetrad. The two vectors of these tetrads, S_α , K_α , are tangent to the orbits of 2D group, L_α is a boundary orthogonal to V_α and to these orbits. These four vectors under comoving relative motion are defined as $S_\alpha = B\delta^2_\beta$, $V_\alpha = -A\delta^1_\beta$, $K_\alpha = C\delta^3_\beta$ and $L_\alpha = A\delta^1_\beta$ alongwith the following constraints

$$S^\alpha S_\alpha = L^\alpha L_\alpha = K_\alpha K^\alpha = 1, \quad V^\alpha V_\alpha = -1, \\ V^\alpha S_\alpha = V^\alpha L_\alpha = K_\alpha V^\alpha = K_\alpha S^\alpha = 0.$$

If the matter configuration within the cylindrical system is charged then, the tensorial formulation describing electromagnetic effects is given by

$$E_{\alpha\beta} = \frac{1}{4\pi} \left(-F^\gamma_\alpha F_{\beta\gamma} + \frac{1}{4} F^{\gamma\delta} F_{\gamma\delta} g_{\alpha\beta} \right), \quad (7)$$

in which $F_{\alpha\beta}$ is a Maxwell field tensor defied in terms of four potential, ϕ_α , as $F_{\alpha\beta} = \phi_{\beta,\alpha} - \phi_{\alpha,\beta}$. The corresponding Maxwell equations of motion are

$$F^{\alpha\beta}_{;\beta} = \mu_0 J^\alpha, \quad F_{[\alpha\beta;\gamma]} = 0, \quad (8)$$

in which J^α is a four current, while μ_0 indicates magnetic permeability. Here we suppose that, due to comoving relativistic motion of the fluid, the electric charge within the matter is at rest which eventually give rise to zero magnetic field. Thus we have

$$\phi_\alpha = \phi(t, r)\delta^0_\alpha, \quad J^\alpha = \rho(t, r)V^\alpha,$$

where ϕ represents scalar potential and ρ indicates the charge density. Equation (8) provide the following equation of motions for charged relativistic cylindrical systems

$$\frac{\partial^2 \phi}{\partial r^2} - \left(\frac{2A'}{A} - \frac{B'}{B} - \frac{C'}{C} \right) \frac{\partial \phi}{\partial r} = 4\pi\rho A^3, \quad (9)$$

$$\frac{\partial}{\partial t} \left(\frac{\partial \phi}{\partial r} \right) - \left(\frac{2\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right) \frac{\partial \phi}{\partial r} = 0, \quad (10)$$

where over dot represents $\frac{\partial}{\partial t}$ while prime indicates $\frac{\partial}{\partial r}$. The solution of Eq.(9) provides

$$\phi' = \frac{A^2 s(r)}{BC},$$

where

$$s(r) = 2\pi \int_0^r \rho ABC dr, \quad (11)$$

is the total charge within cylindrical system of radius r . This quantity can also be evaluated via conservation law of four current, i.e., $J^\mu_{;\mu} = 0$. The electric field strength/intensity that describes charge per unit cylindrical areal surface is written as

$$E(t, r) = \frac{s}{2\pi B}. \quad (12)$$

The $f(R)$ field equations (4) for cylindrically symmetric system coupled with dissipative matter configuration with charged background turn out to be

$$\frac{\dot{C}\dot{B}}{BC} - \frac{C''}{C} - \frac{B''}{B} - \frac{B'C'}{BC} + \alpha_1 = \frac{\kappa A^2}{f_R} \left[\mu + \frac{2\pi E^2}{B^2} + \frac{1}{\kappa} \times \left\{ \frac{f_R''}{A^2} - \frac{\dot{A}f_R'}{A^3} - \frac{A'f_R'}{A^3} - \gamma_1 \right\} \right], \quad (13)$$

$$\left(\frac{B'}{B} + \frac{C'}{C} \right) \frac{\dot{A}}{A} - \frac{\dot{C}}{C} - \frac{\dot{B}}{B} + \left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \frac{A'}{A} = \frac{\kappa}{f_R} [-qA^2 + \frac{1}{\kappa} \left\{ \dot{f}_R' - \frac{A'}{A} f_R - \frac{\dot{A}}{A} f_R' \right\}], \quad (14)$$

$$\frac{B'C'}{BC} - \frac{\ddot{B}}{B} - \frac{\dot{B}\dot{C}}{BC} - \frac{\ddot{C}}{C} + \alpha_1 = \frac{\kappa A^2}{f_R} \left[P_r - \frac{2\pi E^2}{B^2} + \frac{1}{\kappa} \times \left\{ \frac{\ddot{f}_R}{A^2} - \frac{A'f_R'}{A^3} - \frac{\dot{A}f_R'}{A^3} + \gamma_1 \right\} \right], \quad (15)$$

$$\left(\frac{B}{A} \right)^2 \left[\beta_1 + \frac{C''}{C} - \frac{\ddot{C}}{C} \right] = \frac{\kappa B^2}{f_R} \left[P_z + \frac{2\pi E^2}{B^2} + \frac{1}{\kappa} \{ \delta_1 - \frac{C'}{C} \frac{f_R'}{A^2} \} \right], \quad (16)$$

$$\left(\frac{C}{A} \right)^2 \left[\beta_1 + \frac{B''}{B} - \frac{\ddot{B}}{B} \right] = \frac{\kappa C^2}{f_R} \left[P_\phi + \frac{2\pi E^2}{B^2} + \frac{1}{\kappa} \{ \delta_1 \right.$$

$$\left. -\frac{B'}{B} \frac{f'_R}{A^2} \right\}, \quad (17)$$

where

$$\begin{aligned} \alpha_1 &= \left(\frac{\dot{C}}{C} + \frac{\dot{B}}{B} \right) \frac{\dot{A}}{A} + \left(\frac{B'}{B} + \frac{C'}{C} \right) \frac{A'}{A}, \\ \beta_1 &= \frac{\dot{A}^2}{A^2} - \frac{A'^2}{A^2} - \frac{\ddot{A}}{A} + \frac{A''}{A}, \\ \gamma_1 &= \frac{R}{2} \left(\frac{f}{R} - f_R \right) + \left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \frac{f'_R}{A^2} - \left(\frac{B'}{B} + \frac{C'}{C} \right) \frac{f'_R}{A^2}, \\ \delta_1 &= \frac{R}{2} \left(\frac{f}{R} - f_R \right) + \frac{1}{A^2} \left(\ddot{f}_R + \frac{\dot{C}}{C} \dot{f}_R - f''_R \right). \end{aligned}$$

3 MODIFIED STRUCTURE SCALARS

In this section, we shall first notice the effects of dark source $f(R)$ corrections in the expressions of well-known structure scalars. We also consider well-consistent $f(R)$ model and then discuss evolution of these function variables in the dynamics of cylindrically symmetric relativistic systems. Bel (1961) provided the key notion of orthogonal splitting of the Riemann tensor. Later on, Herrera et al. (2009) generalized this concept and discussed many problems related to evolution of compact objects. It is seen that Riemann curvature tensor upon orthogonal decomposition yields set of three tensorial quantities, i.e., $X_{\alpha\beta}$, $Y_{\alpha\beta}$ and $Z_{\alpha\beta}$. The explicit expressions of $Y_{\alpha\beta}$ and $X_{\alpha\beta}$ are carried out by means of trace and traceless components (known as structure scalars) as follows (Herrera, Di Prisco & Ibañez 2011a; Herrera, Di Prisco & Ibáñez 2011b)

$$Y_{\alpha\beta} = \left(S_\alpha S_\beta - \frac{h_{\alpha\beta}}{3} \right) Y_S + \left(K_\alpha K_\beta - \frac{h_{\alpha\beta}}{3} \right) Y_K + \frac{h_{\alpha\beta}}{3} Y_T, \quad (18)$$

$$X_{\alpha\beta} = \left(S_\alpha S_\beta - \frac{h_{\alpha\beta}}{3} \right) X_S + \left(K_\alpha K_\beta - \frac{h_{\alpha\beta}}{3} \right) X_K + \frac{h_{\alpha\beta}}{3} X_T, \quad (19)$$

in which Y_T , Y_S , Y_K , X_S , X_K and X_T are the combinations of the corresponding quantities with respect to standard, charged and effective dark matter configurations. These scalars can be expressed as

$$Y_T = \sum_{i=1}^3 Y_T^{(i)} = T_T^{(G)} + Y_T^{(C)} + Y_T^{(D)}, \quad (20)$$

$$Y_S = \sum_{i=1}^3 Y_S^{(i)} = T_S^{(G)} + Y_S^{(C)} + Y_S^{(D)}, \quad (21)$$

$$Y_K = \sum_{i=1}^3 Y_K^{(i)} = T_K^{(G)} + Y_K^{(C)} + Y_K^{(D)}, \quad (22)$$

$$X_S = \sum_{i=1}^3 X_S^{(i)} = X_S^{(G)} + X_S^{(C)} + X_S^{(D)}, \quad (23)$$

$$X_K = \sum_{i=1}^3 X_K^{(i)} = X_K^{(G)} + X_K^{(C)} + X_K^{(D)}, \quad (24)$$

$$X_T = \sum_{i=1}^3 X_T^{(i)} = X_T^{(G)} + Y_K^{(C)} + X_T^{(D)}. \quad (25)$$

It has been observed that the magnetic part of the Weyl tensor does not vanish for cylindrically symmetric celestial

objects. This gives rise to the birth of another tensorial term $Z_{\alpha\beta}$ Herrera, Di Prisco & Ibáñez (2011b) (after orthogonal decomposition), whose expression is given as

$$Z_{\alpha\beta} = H_{\alpha\beta} + \frac{\kappa}{2f_R} \left(q - \frac{T_{01}^{(D)}}{A^2} \right)^\rho \epsilon_{\alpha\beta\rho}, \quad (26)$$

From this, couple of scalar variables can be found as follows

$$Z_q = \sum_{i=1}^3 Z_q^{(i)} = Z_q^{(G)} + Z_q^{(C)} + Z_q^{(D)}, \quad Z_H = 2H \quad (27)$$

By making use of Eqs.(13)-(17) and (38), above mentioned modified structure scalars are found as

$$Y_T^{(G)} = \frac{\kappa}{2f_R} (\mu + P_z + P_r + P_\phi), \quad Y_T^{(D)} = \frac{2\kappa\pi E^2}{B^2 f_R}, \quad (28)$$

$$Y_T^{(D)} = \frac{\kappa}{2f_R} \left[\frac{1}{A^2} \left(T_{00}^{(D)} + T_{11}^{(D)} \right) + \frac{T_{22}^{(D)}}{B^2} + \frac{T_{33}^{(D)}}{C^2} \right], \quad (29)$$

$$Y_S^{(G)} = \mathbb{E}_S - \frac{\kappa}{2f_R} (P_z - P_r), \quad (30)$$

$$Y_S^{(C)} = -\frac{2\kappa\pi E^2}{B^2 f_R},$$

$$Y_S^{(D)} = \frac{\kappa}{2f_R} \left(\frac{T_{11}^{(D)}}{A^2} - \frac{T_{22}^{(D)}}{B^2} \right), \quad Y_K^{(G)} = \mathbb{E}_K - \frac{\kappa}{2f_R} (P_\phi - P_r) \quad (31)$$

$$Y_K^{(C)} = -\frac{2\pi\kappa}{B^2 f_R} E^2, \quad Y_K^{(D)} = \frac{\kappa}{2f_R} \left(\frac{T_{11}^{(D)}}{A^2} - \frac{T_{33}^{(D)}}{C^2} \right), \quad (32)$$

$$X_S^{(G)} = -\mathbb{E}_S - \frac{\kappa}{2f_R} (P_z - P_r), \quad X_S^{(C)} = -\frac{2\pi\kappa}{B^2 f_R} E^2, \quad (33)$$

$$X_S^{(D)} = \frac{\kappa}{2f_R} \left(\frac{T_{11}^{(D)}}{A^2} - \frac{T_{22}^{(D)}}{B^2} \right), \quad X_K^{(G)} = -\mathbb{E}_K - \frac{\kappa}{2f_R} (P_\phi - P_r) \quad (34)$$

$$X_K^{(C)} = -\frac{2\pi\kappa}{B^2 f_R} E^2, \quad X_K^{(D)} = -\frac{\kappa}{2f_R} \left(\frac{T_{33}^{(D)}}{C^2} - \frac{T_{11}^{(D)}}{A^2} \right), \quad (35)$$

$$X_T^{(G)} = \frac{\kappa}{f_R} (\mu), \quad X_T^{(C)} = \frac{2\pi\kappa}{B^2 f_R} E^2, \quad X_T^{(D)} = \frac{\kappa T_{00}^{(D)}}{2A^2 f_R}, \quad (36)$$

$$Z_q^{(G)} = \frac{\kappa q}{2f_R}, \quad Z_q^{(C)} = 0, \quad Z_q^{(D)} = -\frac{\kappa T_{01}^{(D)}}{2A^2 f_R}, \quad Z_H = 2H, \quad (37)$$

where \mathbb{E}_S , \mathbb{E}_K and H are scalar corresponding to electric and magnetic components of the Weyl tensor (for details see (Herrera, Di Prisco & Ospino 2012)). Their values are

$$\mathbb{E}_S = \frac{1}{A^2} \left(\frac{1}{B^2} C_{0202} - \frac{1}{A^2} C_{0101} \right), \quad (38)$$

$$\mathbb{E}_K = \frac{1}{A^2} \left(\frac{1}{C^2} C_{0303} - \frac{1}{A^2} C_{0101} \right), \quad H = -\frac{1}{C^2 A^2} C_{0313}.$$

The explicit equations for the Weyl tensor components C_{0202} , C_{0101} , C_{0303} , C_{0313} can easily be exhibit with the

help of MATLAB. It is observed from the above formulations that in the modeling of non-static non-rotating diagonal cylindrical systems, one needs to establish eight distinct scalar variables. These scalars are further being demarcated into three portions based on the nature of fluid distribution. It is worthy to mention that these quantities are evaluated by considering general formalism of metric $f(R)$ corrections. One can get GR scalar variables after application of $f(R) = R$ in the above equations.

The conservation laws for standard, charged and effective energy-momentum tensors yield

$$\mu^* + (\mu + P_r)\Theta + q^\alpha a_\alpha + q^\alpha_{;\alpha} + \Pi^{\alpha\beta}\sigma_{\alpha\beta} + \frac{1}{3}\Pi^\alpha_\alpha\Theta + D_0 = 0, \quad (39)$$

$$h^{\alpha\beta}(\Pi^\mu_{\beta;\mu} + P_{r;\beta} + q^*_\beta) + a^\alpha(\mu + P_r) + \frac{4}{3}q^\alpha\Theta - \frac{4\pi E}{A^2 B^2 C} \times (CE' + EC') + q^\mu\sigma^\alpha_\mu + D_1 = 0. \quad (40)$$

where subscript $*$ shows that respective quantity is evaluated under an operator given by $g^* = g_{,\alpha}V^\alpha$, D_0 and D_1 are $f(R)$ dark source terms mentioned in Appendix A, while $\Theta \equiv V^\beta_{;\beta}$ is an expansion scalar given by

$$\Theta = \left(\frac{\dot{B}}{B} + \frac{\dot{A}}{A} + \frac{\dot{C}}{C} \right) \frac{1}{A}.$$

Equation (40) can be recasted as

$$P_r^\dagger + a(\mu + P_r) + q^* - \frac{1}{A} \left[(P_z - P_r) \frac{B'}{B} + (P_\phi - P_r) \frac{C'}{C} \right] - \frac{4\pi E}{A^2 B^2 C} (CE' + EC') - \frac{q}{3}(\sigma_S - 4\Theta + \sigma_K) + D_1 = 0, \quad (41)$$

where subscript \dagger indicates $g^\dagger = g_{,\alpha}L^\alpha$ operator, while σ_K and σ_S are shear scalars, whose values are

$$\sigma_K = \left(\frac{\dot{C}}{C} - \frac{\dot{A}}{A} \right) \frac{1}{A}, \quad \sigma_S = \left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A} \right) \frac{1}{A}.$$

In order to present $f(R)$ theory of gravity as an acceptable theory, one should consider a viable as well as well-consistent $f(R)$ model. It was one of the challenging tasks of relativistic astrophysicists to introduce a model that can not only explain dark energy epochs but also discuss inflationary eras of cosmos, with the single formulation. To generate such kind of results from $f(R)$ gravity models, they must obey certain restrictions imposed by terrestrial and solar system experiments with relativistic background (Nojiri & Odintsov 2007a). Here, we consider a well-known realistic $f(R)$ model that tends to unify inflation and dark energy eras. Its mathematical formulation is given as follows

$$f(R) = \frac{\alpha R^{2n} - \beta R^n}{1 + \gamma R^n}. \quad (42)$$

where n is a non-negative and non-zero integer while α , β and γ are positive constants. It is found that this model produces instabilities-free $f(R)$ contribution in the explanation of cosmological evolution. It yields several results consistent with the outcomes noticed in cosmological and solar system tests. Thus, this may be considered as a viable model in which late-time cosmic acceleration as well as early-time inflationary eras are naturally unified within single model.

The two very important expressions that would be helpful in the description of evolutionary phases of radiating

anisotropic cylindrically symmetric systems can be obtained after using Eqs.(13)-(17), (20)-(25) and (27). These expressions were firstly computed by Herrera, Di Prisco & Ospino (2012) in GR. These, in Maxwell- $f(R)$ gravity, turn out to be

$$\begin{aligned} & \frac{\kappa(1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \left(2\mu + \frac{2\pi E^2}{B^2} \right. \\ & + \frac{n\gamma R^{2n}(\alpha R^n - \beta)}{2(1 + \gamma R^n)^2} + \frac{\alpha R^{2n}(1 - 2n) - \beta R^n(1 - n)}{2(1 + \gamma R^n)} + P_z \\ & + P_r + P_\phi + \frac{2\psi_{tt}}{A^2} + \frac{2\psi_{rr}}{A^2} + \frac{\psi_{zz}}{B^2} + \frac{\psi_{\phi\phi}}{C^2} \Big)^\dagger + \left(\mu + P_r + \frac{\psi_{tt}}{A^2} \right. \\ & + \frac{\psi_{rr}}{A^2} \Big) \frac{3a\kappa(1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \\ & - \frac{2\kappa(1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \left(q - \frac{\psi_{tr}}{A^2} \right) \\ & \times (\sigma_S - \Theta + \sigma_K) + \frac{3\kappa(1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \\ & \times \left(q - \frac{\psi_{tr}}{A^2} \right)^* = 3(X_K - Y_K) \frac{C'}{AC} - (Y_S + Y_K - X_S - X_K)^\dagger \\ & + 3(X_S - Y_S) \frac{B'}{AB} - 6H(\sigma_S - \sigma_K), \end{aligned} \quad (43)$$

$$\begin{aligned} & - \frac{\kappa(1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} (\mu - P_r \\ & + \frac{6\pi E^2}{B^2} - P_z - \frac{\alpha R^{2n}(1 - 2n) - \beta R^n(1 - n)}{2(1 + \gamma R^n)} + \frac{n\gamma R^{2n}(\alpha R^n - \beta)}{2(1 + \gamma R^n)^2} \\ & + 2P_\phi + \frac{\psi_{tt}}{A^2} - \frac{\psi_{rr}}{A^2} - \frac{\psi_{zz}}{B^2} + \frac{2\psi_{\phi\phi}}{C^2} \Big)^\dagger - \left(q - \frac{\psi_{tr}}{A^2} \right) \\ & \frac{\kappa(1 + \gamma R^n)^2(\sigma_S - \Theta - 2\sigma_K)}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \\ & - \frac{3\kappa(1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \\ & \times \left(P_\phi - P_r - \frac{\psi_{rr}}{A^2} + \frac{\psi_{\phi\phi}}{C^2} \right) \frac{C'}{AC} = 6H^* + (2Y_S - 2X_S - Y_K \\ & + X_K)^\dagger + 3(Y_S - X_S) \frac{B'}{AB} + 3a(X_K - X_S - Y_K + Y_S) \\ & + 6H(\Theta - \sigma_K), \end{aligned} \quad (44)$$

in which a is a scalar term expressed in terms of four acceleration as $a_\beta = aL_\beta$. Its value is $a = \frac{\dot{A}}{A^2}$. The quantities $\psi_{\alpha\beta}$ are $f(R)$ corrections and can easily be evaluated after using Eq.(42) in the following equations

$$\psi_{\alpha\beta} = T^{(D)}_{\alpha\beta} - \frac{1}{2}(f - Rf_R)g_{\alpha\beta}.$$

The evolution and stability of relativistic self-gravitating systems during gravitational collapse have become interesting phenomenon not only in GR but also in modified gravity theories. The utmost relevance of collapsing mechanism lies at the center of stellar structure formation which happens when smooth initial matter configurations will ultimately collapse. This process gives rise to overwhelm structures such as stars, stellar groups and planets. A system begins collapsing once it experiences an inhomogeneous stellar state. Which factors are responsible for the emergence of inhomogeneity phases in the initially homogeneous system? To answer this, we have explored above two

relations. Penrose & Hawking (2007) claimed that system's inhomogeneity factors are very important in the discussion of its collapsing behavior. So, they explored irregularities in the energy density of spherical relativistic stars by means of Weyl scalar. In order to see how above relations work, we consider cylindrical isotropic non-radiating system (simplest case). Under this scenario, Eqs.(21)-(24) give

$$\begin{aligned} Y_S &= \mathbb{E}_S + \frac{\kappa}{2f_R} \left(\frac{T_{11}^{(D)}}{A^2} - \frac{4\pi E^2}{B^2} - \frac{T_{22}^{(D)}}{B^2} \right), \\ Y_K &= \mathbb{E}_K + \frac{\kappa}{2f_R} \left(\frac{T_{11}^{(D)}}{A^2} - \frac{4\pi}{B^2} E^2 - \frac{T_{33}^{(D)}}{C^2} \right), \\ X_S &= -\mathbb{E}_S + \frac{\kappa}{2f_R} \left(\frac{T_{11}^{(D)}}{A^2} - \frac{4\pi}{B^2} E^2 - \frac{T_{22}^{(D)}}{B^2} \right), \\ X_K &= -\mathbb{E}_K + \frac{\kappa}{2f_R} \left(\frac{T_{33}^{(D)}}{C^2} - \frac{4\pi}{B^2} E^2 - \frac{T_{11}^{(D)}}{A^2} \right), \end{aligned}$$

Using these relations along with Eq.(42) in Eq.(44), we get

$$\begin{aligned} & - \frac{\kappa(1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} (\mu \\ & + \frac{6\pi E^2}{B^2} - \frac{\alpha R^{2n}(1 - 2n) - \beta R^n(1 - n)}{2(1 + \gamma R^n)} + \frac{n\gamma R^{2n}(\alpha R^n - \beta)}{2(1 + \gamma R^n)^2} \\ & + \frac{\psi_{tt}}{A^2} - \frac{\psi_{rr}}{A^2} - \frac{\psi_{zz}}{B^2} + \frac{2\psi_{\phi\phi}}{C^2})^\dagger - \left(\frac{\psi_{\phi\phi}}{C^2} - \frac{\psi_{rr}}{A^2} \right) \frac{C'}{AC} \\ & \times \frac{3\kappa(1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \\ & = 6H^* - 2 \left\{ \left(\frac{\psi_{zz}}{C^2} - \frac{\psi_{rr}}{A^2} + 2\mathbb{E}_S \right) \right. \\ & \times \frac{\kappa(1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \Big\}^\dagger \\ & + 6\mathbb{E}_S \frac{B'}{AB} + 6H(\Theta - \sigma_K) - 6a \left(\frac{\psi_{zz}}{C^2} - \frac{\psi_{rr}}{A^2} + 2\mathbb{E}_S \right) \\ & \times \frac{\kappa(1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]}, \quad (45) \end{aligned}$$

For conformally flat solutions. we have $\mathbb{E}_S = 0 = H$, then μ^\dagger is directly related with electromagnetic field and $f(R)$ dark source terms. In GR, above equation reduces to

$$\mu^\dagger = 0 \Rightarrow \mu = \mu(t).$$

This shows that in GR, it is easy for the system to enter or leave the homogeneous phase. However, in Maxwell- $f(R)$ gravity, electric charge as well as $f(R)$ extra curvature terms tend to produce hindrances for the system to leave and enter homogeneous state. Thus, it is relatively difficult for the system to enter in the unstable phases in Maxwell- $f(R)$ field, thereby indicating that more stable cylindrical configurations of stellar systems exists in modified gravity. Similar result can be deduced from Eq.(43).

In the framework of $f(R)$ model given in Eq.(42), modified versions of scalar quantities are also evaluated and are mentioned in Appendix A. We know that these scalar variables are being explored after orthogonal decomposition of Reimann curvature tensors. It has been analyzed from the work of many researchers

that structure scalars have a crucial importance in discussing the dynamical evolution of self-gravitating spherical celestial bodies (Herrera, Di Prisco & Ibáñez 2011b; Herrera, Di Prisco & Ospino 2012; Sharif & Yousaf 2014b, 2015a,b). We shall now discuss the importance of these variables in the modeling of relativistic cylindrical stellar object in the coming sections.

4 $F(R)$ CYLINDRICAL GENERALIZATION TO SPHERICAL MISNER-SHARP MASS FUNCTION

This section is aimed to investigate thermoinertial effects on the modified effective inertial mass of locally anisotropic non-rotating cylindrical celestial systems in the presence of electromagnetic field. We also discuss the influence of heat radiating parameter in the subsequent evolution of collapsing stellar objects in the Jordan frame of metric $f(R)$ gravity. For this, we first evaluate modified version of dynamical-transport equation and then formulate mass function for charged cylindrical relativistic interior with inflationary and late time cosmic accelerating $f(R)$ model. We shall express these terminologies in terms of modified versions of structure scalars. The velocity of the collapsing fluid is defined as the change in cylindrical areal radius corresponding to proper time. This velocity is often taken to be less than zero for collapsing relativistic interior and is given as

$$U = \frac{\dot{C}}{A} = C^* < 0 \text{ (for collapsing system)}, \quad (46)$$

which after using 11 field equation gives

$$\begin{aligned} U^* &= \frac{aC'}{A} - \frac{\kappa C(1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \\ & \times \left(P_r - \frac{2\pi E^2}{B^2} + \frac{\psi_{rr}}{A^2} \right) - \frac{\alpha R^{2n}(1 - 2n) - \beta R^n(1 - n)}{2(1 + \gamma R^n)} \\ & + \frac{n\gamma R^{2n}(\alpha R^n - \beta)}{2(1 + \gamma R^n)^2} - \frac{C}{A^2} \left[\frac{\ddot{B}}{B} + \frac{\dot{B}}{B} \left(\frac{\dot{C}}{C} - \frac{\dot{A}}{A} \right) - \frac{B'}{B} \right. \\ & \times \left. \left(\frac{C'}{C} + \frac{A'}{A} \right) \right]. \end{aligned}$$

This equation can also be manipulated as

$$\begin{aligned} U^* &= - \frac{\kappa C(1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \\ & \times \left(P_r - \frac{2\pi E^2}{B^2} + \frac{\psi_{rr}}{A^2} \right) + \frac{aC'}{A} - \frac{\alpha R^{2n}(1 - 2n) - \beta R^n(1 - n)}{2(1 + \gamma R^n)} \\ & + \frac{n\gamma R^{2n}(\alpha R^n - \beta)}{2(1 + \gamma R^n)^2} - \frac{C}{B^2} \left(\frac{R_{2323}}{C^2} - \frac{R_{0202}}{A^2} \right), \quad (47) \end{aligned}$$

where R_{2323} and R_{0202} are components of curvature tensor and are written down as

$$\begin{aligned} R_{2323} &= B^2 \left(\frac{A'B'}{AB} + \frac{\dot{A}\dot{B}}{AB} - \frac{\ddot{B}}{B} \right), \\ R_{0202} &= -\frac{C^2}{A^2} \left(\frac{B'C'}{BC} - \frac{\dot{B}\dot{C}}{BC} \right) B^2. \end{aligned}$$

Solving Eq.(47) for acceleration scalar, a , using this solution in second dynamical equation (41), we get

$$U^*(\mu + P_r)$$

$$\begin{aligned}
 = & - \left[\left\{ \frac{D_3 C'}{A} + \frac{\kappa C(1 + \gamma R^n)^2 (\mu + P_r)}{n R^{n-1} [(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n (\alpha R^n - \beta)]} + \frac{(1 + \gamma R^n)^2}{n R^{n-1} [(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n (\alpha R^n - \beta)]} \right\} - \frac{C}{3} (\mathbb{E}_S \right. \\
 \times & \left(-\frac{2\pi E^2}{B^2} - \frac{\alpha R^{2n}(1 - 2n) - \beta R^n(1 - n)}{2(1 + \gamma R^n)} + \frac{n\gamma R^{2n}(\alpha R^n - \beta)}{2(1 + \gamma R^n)^2} - 2\mathbb{E}_K \right) + \frac{\kappa C(1 + \gamma R^n)^2}{6n R^{n-1} [(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n (\alpha R^n - \beta)]} \\
 & \left. + P_r + \frac{\psi_{rr}}{A^2} \right\} - \frac{A}{B^2} \times (\mu + P_r) \left(\frac{R_{0202}}{A^2} - \frac{R_{2323}}{C^2} \right) \frac{C'}{C} \Bigg] \\
 & \times \left(\mu - P_\phi + \frac{6\pi}{B^2} E^2 - P_r + 2P_z + \frac{\psi_{tt}}{A^2} - \frac{\psi_{rr}}{A^2} + \frac{2\psi_{zz}}{B^2} \right. \\
 & \left. + \left[-q^* + \frac{1}{3}(\sigma_K + \sigma_S - 4\Theta)q \right] \frac{C'}{A} - \frac{C'}{A} \left[P_r^\dagger - \frac{1}{A} \left(\frac{B'}{B} (P_z - P_r) - \frac{\psi_{\phi\phi}}{C^2} + \frac{n\gamma R^{2n}(\alpha R^n - \beta)}{2(1 + \gamma R^n)^2} + \frac{\alpha R^{2n}(1 - 2n) - \beta R^n(1 - n)}{2(1 + \gamma R^n)} \right) \right. \right. \\
 & \left. \left. + \frac{C'}{C} (P_\phi - P_r) \right) + \frac{4\pi E}{A^2 B^2 C} (EC' + E'C) \right], \quad (48)
 \end{aligned}$$

where D_3 represents $f(R)$ extra degrees of freedom. One can obtain this expression by using Eq.(42) in D_1 . Left hand side (LHS) is formed through the product of time derivative of matter four velocity and inertial mass (density). Nevertheless, right hand side (RHS) of the above equation can be considered as a combination of three different portions (in three square brackets). The first square bracket encapsulates effective portion of relativistic gravitational force (along with inflationary and late time accelerating $f(R)$ corrections), while the second part describes parameters controlling dissipative mechanism of the charged cylindrical collapsing systems. The last term leads to hydrodynamic force as it is the combination of pressure and electromagnetic anisotropic gradient entities for evolving cylindrically symmetric stellar object. It is worthy to stress that hydrodynamic era can be referred as matter particles that not necessarily radiate/transport heat. Equation (48) can be rewritten in Newtonian form as

$$Acceleration \times Mass \text{ density} = Force.$$

Now we calculate charged cylindrical mass function m framed within $f(R)$ background. The couple of Reimann tensor components, R_{0202} and R_{2323} can be written by means of metric $f(R)$ structure functions as

$$\frac{B^2}{3} (2Y_S + Y_T - X_T + X_S + Y_K - X_K) = \frac{1}{A^2} R_{0202} - \frac{1}{C^2} R_{2323}.$$

Using Eqs.(20)-(25) and feeding back the values of scalar variables from Eqs.(A3)-(A16), we can modify above equation as

$$\begin{aligned}
 \frac{1}{A^2} R_{0202} - \frac{1}{C^2} R_{2323} = & \frac{B^2}{3} (\mathbb{E}_S - 2\mathbb{E}_K) + \frac{\kappa B^2}{3f_R} (2P_r - P_z \\
 & - \frac{6\pi}{B^2} E^2 + \frac{\alpha R^{2n}(1 - 2n) - \beta R^n(1 - n)}{(1 + \gamma R^n)} + \frac{n\gamma R^{2n}(\alpha R^n - \beta)}{(1 + \gamma R^n)^2} \\
 & - \frac{\mu}{2} - \frac{\psi_{zz}}{B^2} - \frac{\psi_{tt}}{2A^2} + \frac{P_\phi}{2} + \frac{2\psi_{rr}}{A^2} + \frac{\psi_{\phi\phi}}{2C^2} \Bigg). \quad (49)
 \end{aligned}$$

Equations (48) and (49) simultaneously provide

$$\begin{aligned}
 & \frac{\kappa C(1 + \gamma R^n)^2}{n R^{n-1} [(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n (\alpha R^n - \beta)]} \\
 & \left(P_r - \frac{2\pi}{A^2} E^2 + \frac{\psi_{rr}}{A^2} + \frac{\alpha R^{2n}(1 - 2n) - \beta R^n(1 - n)}{2(1 + \gamma R^n)} \right. \\
 & \left. + \frac{n\gamma R^{2n}(\alpha R^n - \beta)}{2(1 + \gamma R^n)^2} \right) - \frac{C}{B^2} \left(\frac{1}{A^2} R_{0202} - \frac{1}{C^2} R_{2323} \right) \\
 & = \frac{\kappa C(1 + \gamma R^n)^2}{n R^{n-1} [(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n (\alpha R^n - \beta)]} \\
 & \left(P_r - \frac{2\pi}{B^2} E^2 + \frac{\psi_{rr}}{A^2} \right.
 \end{aligned}$$

The above relation help us to calculate possible modification of Misner-Sharp mass formulation (Misner & Sharp 2012) for cylindrical collapsing object in the presence of electromagnetic field and $f(R)$ extra curvature terms. For this purpose, we first calculate Eqs.(47) and (55) mentioned in Herrera et al. (2004) and Di Prisco et al. (2007), respectively in Maxwell- $f(R)$ gravity. Then, we compare these results with Eq.(50), keeping in mind that regenerative pressure gradient has the same contribution as in spherical symmetrical stellar systems. Consequently, we found the following peculiar form of mass function

$$\begin{aligned}
 m = & \frac{\kappa C^3(1 + \gamma R^n)^2}{6n R^{n-1} [(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n (\alpha R^n - \beta)]} \\
 & \times \left(\mu + \frac{6\pi}{B^2} E^2 - \frac{\psi_{rr}}{A^2} + \frac{\psi_{tt}}{A^2} - P_\phi + \frac{2\psi_{zz}}{B^2} \right. \\
 & - \frac{\alpha R^{2n}(1 - 2n) - \beta R^n(1 - n)}{2(1 + \gamma R^n)} + \frac{n\gamma R^{2n}(\alpha R^n - \beta)}{2(1 + \gamma R^n)^2} \\
 & \left. - P_r + 2P_z - \frac{\psi_{\phi\phi}}{C^2} \right) - \frac{C^3}{3} (\mathbb{E}_S - 2\mathbb{E}_K). \quad (51)
 \end{aligned}$$

We can obtain from Eqs.(20)-(25), (27), (A3)-(A16) and above equation

$$\begin{aligned}
 \frac{3m}{C^3} = & \frac{(1 + \gamma R^n)^2}{2n R^{n-1} [(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n (\alpha R^n - \beta)]} \\
 & \times \left(\mu + P_\phi + \frac{10\pi}{B^2} E^2 - 2P_r + P_z - \frac{2\psi_{rr}}{A^2} \right. \\
 & - \frac{\alpha R^{2n}(1 - 2n) - \beta R^n(1 - n)}{(1 + \gamma R^n)} + \frac{n\gamma R^{2n}(\alpha R^n - \beta)}{(1 + \gamma R^n)^2} \\
 & \left. + \frac{\psi_{tt}}{A^2} + \frac{\psi_{\phi\phi}}{C^2} + \frac{\psi_{zz}}{B^2} \right) - (Y_S - 2Y_K), \quad (52)
 \end{aligned}$$

$$\begin{aligned}
 \frac{3m}{C^3} = & \frac{(1 + \gamma R^n)^2}{2n R^{n-1} [(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n (\alpha R^n - \beta)]} \\
 & \times \left(\mu + 3P_z + \frac{2\pi}{B^2} E^2 - 3P_\phi + \frac{\psi_{tt}}{A^2} - \frac{3\psi_{\phi\phi}}{C^2} + \frac{2\psi_{zz}}{B^2} \right) \\
 & + (X_S - 2X_K). \quad (53)
 \end{aligned}$$

In the presence of electromagnetic field and several scalars variables, the cylindrical mass function is obtained after using Eqs.(20)-(25), (27), (A3)-(A16), (41), (43) and (44) as

$$\begin{aligned}
 & - \left[\frac{\kappa(1 + \gamma R^n)^2}{n R^{n-1} [(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n (\alpha R^n - \beta)]} \right. \\
 & \times \left(\mu + \frac{2\pi}{B^2} E^2 - \frac{\alpha R^{2n}(1 - 2n) - \beta R^n(1 - n)}{2(1 + \gamma R^n)} \right. \\
 & \left. - \frac{n\gamma R^{2n}(\alpha R^n - \beta)}{2(1 + \gamma R^n)^2} + \frac{\psi_{tt}}{A^2} \right) \Bigg]^\dagger + (X_S + 3Y_S + X_K - 3Y_K)^\dagger \\
 & = \frac{(1 + \gamma R^n)^2}{n R^{n-1} [(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n (\alpha R^n - \beta)]} \left(q - \frac{\psi_{tr}}{A^2} \right)
 \end{aligned}$$

$$\begin{aligned}
& (\sigma_K - \Theta - 2\sigma_S) - (Y_S + X_S) \frac{3B'}{AB} + \frac{12\kappa\pi E}{A^2 B^2 C f_R} \left\{ E' C + C' \right. \\
& \times \left(E - \frac{k}{C} \right) \left. \right\} + (Y_K - X_K) \frac{3C'}{AC} - 6H^* + 6H(\sigma_S - \Theta) \\
& - 3a(Y_S - Y_K - X_S + X_K). \quad (54)
\end{aligned}$$

From Eqs.(52) and (53), we have

$$\begin{aligned}
\frac{6m}{C^3} &= \frac{\kappa(1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \\
&\times \left(\mu + \frac{2\pi}{B^2} E^2 - \frac{\alpha R^{2n}(1 - 2n) - \beta R^n(1 - n)}{2(1 + \gamma R^n)} + \frac{\psi_{tt}}{A^2} \right. \\
&\left. - \frac{n\gamma R^{2n}(\alpha R^n - \beta)}{2(1 + \gamma R^n)^2} \right) - 3Y_S - X_K + 3Y_K - X_S. \quad (55)
\end{aligned}$$

Applying \dagger on both sides of Eq.(55) and taking into account Eq.(54), we obtain

$$\begin{aligned}
\left(\frac{6m}{C^3} \right)^\dagger &= \\
&- \frac{4\pi\kappa E(1 + \gamma R^n)^2}{nA^2 B^2 C R^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \\
&\times \left\{ EC'(1 - 2\pi) + C \left(E' - \frac{4\pi EB'}{B} \right) \right\} \frac{3(Y_S + X_S)}{AB} B' \\
&+ 6H^* - 6H(\sigma_S - \Theta) + \frac{3(X_K - Y_K)}{CA} C' \\
&+ \frac{\kappa(1 + \gamma R^n)^2(2\sigma_S - \sigma_K + \Theta)}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \\
&\times \left(q - \frac{\psi_{tr}}{A^2} \right) + 3a(X_K + Y_S - X_S - Y_K),
\end{aligned}$$

that yields

$$\begin{aligned}
m &= \\
\frac{C^3}{2} \int &\left[-\frac{4\pi\kappa E(1 + \gamma R^n)^2}{nA^2 B^2 C R^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \right. \\
&\times \left\{ EC'(1 - 2\pi) + C \left(E' - \frac{4\pi EB'}{B} \right) \right\} + \frac{(Y_S + X_S)}{B} B' \\
&- 2AH(\sigma_S - \Theta) + 2AH^* + \frac{(X_K - Y_K)}{C} C' \\
&+ \frac{\kappa A(1 + \gamma R^n)^2(2\sigma_S + \Theta - \sigma_K)}{3nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \left(q \right. \\
&\left. - \frac{\psi_{tr}}{A^2} \right) + aA(Y_S - X_S + X_K - Y_K) \left. \right] dr + \frac{C^3 \zeta(t)}{6}, \quad (56)
\end{aligned}$$

in which ζ is an integration function.

One can easily observe from Eq.(44) that if our system, that is coupled with non-radiating isotropic matter configurations, is conformally flat, then the homogeneity in the energy density is controlled by $f(R)$ degrees of freedom. Further, if one takes current cosmological value of curvature tensor, then system will encapsulate regular energy density throughout over the celestial object with conformally flat environment. At that case, the last portion of the above equation makes the mass function for the regular configurations of celestial cylindrical object. The present version of mass function (mentioned at Eq.(57)) points out that how much system scalar variables and electric charge are important in its formulation. Substituting X_S , Y_K , Y_S , X_K in Eq.(56), we get

$$m = \frac{C^3}{2} \int \left[\frac{-\kappa(1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \right.$$

$$\begin{aligned}
&\times \left(P_z - P_r + \frac{4\pi}{B^2} E^2 + \frac{\psi_{zz}}{B^2} - \frac{\psi_{rr}}{A^2} \right) \frac{B'}{B} - \{ EC'(1 - 2\pi) \\
&+ C \left(E' - \frac{4\pi EB'}{B} \right) \} \\
&\times \frac{4\pi\kappa E(1 + \gamma R^n)^2}{nA^2 B^2 C R^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \\
&- 2[H(\sigma_S - \Theta) - H^*] A - 2\mathbb{E}_K \frac{C'}{C} \\
&+ \frac{\kappa A(1 + \gamma R^n)^2}{3nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \\
&\times \left(q + \frac{\psi_{tr}}{A^2} \right) (2\sigma_S + \Theta - \sigma_K) \left. \right] dr + \frac{C^3 \zeta(t)}{6}. \quad (57)
\end{aligned}$$

in which influence of tidal forces, dark source extra curvature terms, radiating parameters, local anisotropic pressure, shear, expansion and modified version of scalar functions can easily be exhibited. It can be observed that for conformally flat isotropic non-dissipative cylinder, the inhomogeneity in the energy density is disturbed by electric charge and realistic $f(R)$ corrections. Then last term of the above equation holds fundamental importance in the measurement of mass function of the uncharged perfect cylindrical compact object with $f(R)$ corrections.

The problem of collapsing systems can be well discussed by joining two appropriate geometries of interior and exterior spacetimes. For this purpose, we consider that system relativistic motion is characterized by three dimensional timelike surface represented by Ω . Thus, Ω demarcated our manifold into interior and exterior portions, denoted, respectively, by \mathcal{V}^- and \mathcal{V}^+ . The spacetime for \mathcal{V}^+ is Chao-Guang (1995)

$$ds_+^2 = \left(-\frac{2M}{r} \right) d\nu^2 + 2d\nu dr - r^2(d\phi^2 + \zeta^2 dz^2), \quad (58)$$

where M is a cylindrical gravitating mass and ζ indicates arbitrary constant. The spacetime for \mathcal{V}^- is given in Eq.(5). For continuous matching of \mathcal{V}^- and \mathcal{V}^+ , we use Darmois (1927) and Senovilla (2013) matching criteria. Darmois conditions require continuity of both line elements and extrinsic curvature over Ω (see for details Chan (2011)), while Senovilla matching conditions requires

$$R|_+^\pm = 0, \quad f_{,RR}[\partial_\nu R]_+^\pm = 0, \quad f_{,RR} \neq 0. \quad (59)$$

These couple of constraints assert that Ricci invariant must be continuous over Ω even for matter thin shells.

4.1 Gravity Induced by Specific Versions of $f(R)$ Models

Using Senovilla (2013) and Darmois (1927) criteria over the hypersurface in Eq.(57), we found some relationships between mass and radius for different values of parameters mentioned in (42) $f(R)$ model. These parameters allow us to set up various cylindrical compact models controlled by some specific $f(R)$ configurations.

Model 1. First, we consider the simple form of Eq.(42) given by

$$f(R) = \alpha R^2, \quad (60)$$

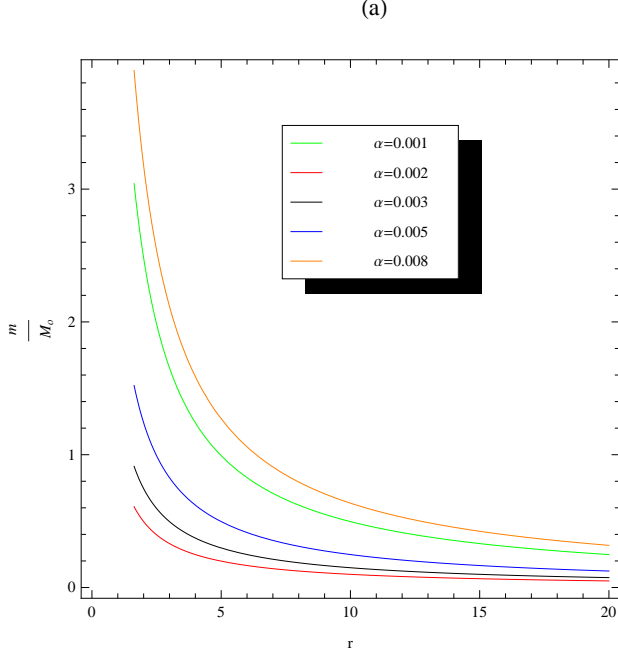


Figure 1. Mass radius relationship diagram for relativistic cylinders in $f(R)$ gravity, with different values of α along with $n = 1$, $\beta = 0 = \gamma$.

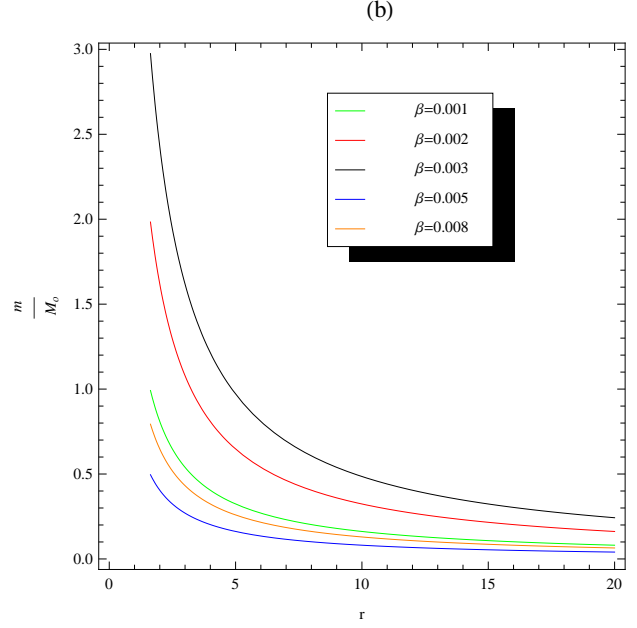


Figure 3. Mass radius relationship diagram for relativistic cylinders in $f(R)$ model, with different values of β along with $n = 1$, $\gamma = 0$.

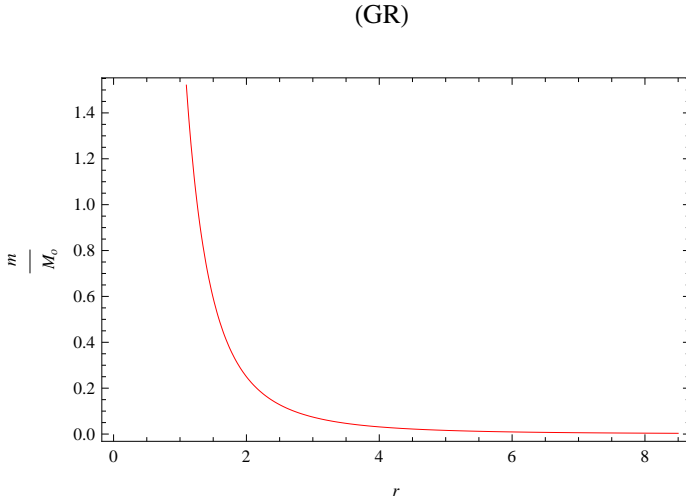


Figure 2. Mass radius relationship diagram in GR.

where α is positive. This type of model was claimed to describe early time cosmic inflation due to an additional αR^2 term. These corrections in the field equations (Felice & Tsujikawa 2010) are consistent with temperature anisotropies noticed in cosmic microwave background. Now, we shall see the effects of $f(R)$ corrections in the dynamical behavior of cylindrical anisotropic matter distribution. We found mass radius relationships with different values of α and $\beta = \gamma = 0$ mentioned in figure 1. It is seen that the maximum $\frac{m}{M_\odot}$ ratio of relativistic cylinder is about 3.4 against 1.1 value of radius. In GR, we also found mass radius relationship and concluded that the maximum $\frac{m}{M_\odot}$ ratio is 1.4 for 1.1 value of radius as seen in figure 2. Thus, such

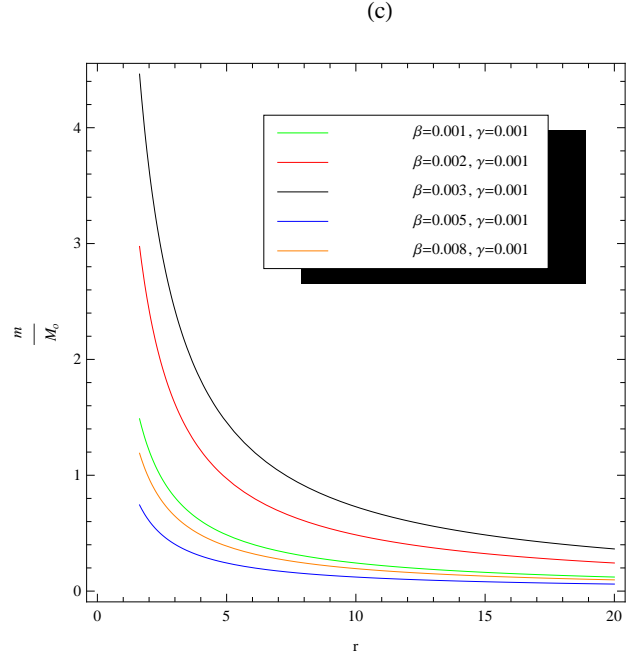


Figure 4. Mass radius relationship diagram for relativistic cylinders in $f(R)$ model, with different values of β and γ along with $n = 1$.

modified models yield more compact relativistic cylinders with smaller radii than GR. Our this result supports the analysis of Astashenok, Capozziello & Odintsov (2011).

Model 2. It would be interesting to analyze $f(R)$ corrections of the following type

$$f(R) = \alpha R^2 - \beta R, \quad (61)$$

where α and β are positive. The models of this type could allow to deal with some unexplored cylindrical self-gravitating dissipative systems that was escaped from standard gravitational theory. We found mass radius relationship by considering above mentioned corrections in the action. We conclude that for different values of β with fixed value of α and $\gamma = 0$, the maximum $\frac{m}{M_\odot}$ ratio of cylindrical stellar objects is 3.0 for about 1.6 units of radius as shown in figure 3. This indicates that very massive stellar compact objects that could not be obtained as solutions of the standard stellar structures theory could be achieved in the realm of above mentioned $f(R)$ gravity.

Model 3. Now, we consider the following specific form of $f(R)$ correction in the action

$$f(R) = \frac{\alpha R^2 - \beta R}{1 + \gamma R}, \quad (62)$$

where α , β and γ are positive constants. Here, we found mass radius relation diagram (figure 4) for different values of β and γ with fixed value of α . We conclude maximum $\frac{m}{M_\odot}$ ratio to be 4.5 against about 1.5 units of radius. This $\frac{m}{M_\odot}$ ratio is found to be greater than that obtained in the above couple of $f(R)$ models. Further, it can be analyzed from figures 2-4 that wide range of massive compact objects occur in $f(R)$ gravity. This fact can be interpreted as there is a possibility of the occurrence of more massive stable cylindrical objects in modified gravity (with some specific $f(R)$ models). These results support the consequences of Farinelli et al. (2014).

5 $F(R)$ DYNAMICAL-TRANSPORT EQUATION

The transport expression from casual radiating theory is

$$\tau q^* = -\frac{1}{2}\eta q \Omega^2 \left(\frac{\tau}{\eta \Omega^2} \right)^* - \eta(\Omega^\dagger + \Omega a) - q - \frac{1}{2}\tau q \Theta,$$

where τ indicates relaxation time, η is a thermal conductivity, while Ω stands for temperature. On setting $\tau = 0$ in the above equation, one can get Eckart-Landau equation (Eckart 1940; Landau & Lifshitz 1940). The non-vanishing independent component is

$$\tau q^* + q = -\eta(\Omega^\dagger + \Omega a) - \frac{1}{2}\eta \Omega^2 q \left(\frac{\tau}{\eta \Omega^2} \right)^* - \frac{1}{2}q \tau \Theta. \quad (63)$$

After using Eqs.(50), (51) and (63) in (48), we get

$$\begin{aligned} (\mu + P_r) \left\{ 1 - \frac{\eta \Omega}{\tau(\mu + P_r)} \right\} U^* = & - \left\{ \left(P_r + \frac{\alpha R^{2n}(1-2n)}{2(1+\gamma R^n)} \right. \right. \\ & - \frac{\beta R^n(1-n)}{2(1+\gamma R^n)} + \frac{n\gamma R^{2n}(\alpha R^n - \beta)}{2(1+\gamma R^n)^2} - \frac{2\pi}{A^2} E^2 + \frac{T_{11}^{(D)}}{A^2} \Big) \\ & \times [\kappa C^3(1+\gamma R^n)^2][2nR^{n-1}[(1+\gamma R^n)(2\alpha R^n - \beta) - \gamma R^n \\ & \times (\alpha R^n - \beta)]]^{-1} + m \Big\} \frac{(\mu + P_r)}{C^2} \left\{ 1 - \frac{\eta \Omega}{\tau(\mu + P_r)} \right\} + \frac{4\pi E}{A^3 B^2 C} \\ & \times (CE' + EC')C' + \frac{C'}{A} \left[-P_r^\dagger + \frac{1}{A} \left\{ (P_\phi - P_r) \frac{C'}{C} + (P_z \right. \right. \\ & \left. \left. - P_r) \frac{B'}{B} \right\} \right] + \frac{C'}{A} \left[q \left\{ \frac{1}{\tau} + \frac{\eta \Omega^2}{2\tau} \left(\frac{\tau}{\eta \Omega^2} \right)^* + \frac{(\sigma_S + \sigma_K)}{3} \right. \right. \end{aligned}$$

$$\left. - \frac{5}{6}\Theta \right\} + \frac{\eta \Omega^\dagger}{\tau} \Big] - \frac{D_3 C'}{A},$$

which can be recast as

$$\begin{aligned} (\mu + P_r)(1 - \mathfrak{a})U^* = & F_{grav}(1 - \mathfrak{a}) + \frac{C'}{A} \left[-P_r^\dagger + \frac{1}{A} \{ (P_\phi \right. \\ & \left. - P_r) \frac{C'}{C} + (P_z - P_r) \frac{B'}{B} \} - D_1 \right] + \frac{C'}{A} \left[q \left\{ \frac{1}{\tau} + \frac{\eta \Omega^2}{2\tau} \right. \right. \\ & \times \left(\frac{\tau}{\eta \Omega^2} \right)^* + \frac{(\sigma_S + \sigma_K)}{3} - \frac{5}{6}\Theta \Big\} + \frac{\eta \Omega^\dagger}{\tau} \Big] + \frac{4\pi E}{A^3 B^2 C} \\ & \times (CE' + EC')C', \end{aligned} \quad (64)$$

where

$$\begin{aligned} F_{grav} = & -\frac{(\mu + P_r)}{C^2} \left\{ \left(P_r + \frac{\alpha R^{2n}(1-2n) - \beta R^n(1-n)}{2(1+\gamma R^n)} \right. \right. \\ & \left. \left. + \frac{n\gamma R^{2n}(\alpha R^n - \beta)}{2(1+\gamma R^n)^2} - \frac{2\pi}{A^2} E^2 + \frac{T_{11}^{(D)}}{A^2} \right) \right. \\ & \times \frac{[\kappa C^3(1+\gamma R^n)^2]}{[2nR^{n-1}[(1+\gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]]} + m \Big\}, \\ \mathfrak{a} = & \frac{\eta \Omega}{\tau(\mu + P_r)}. \end{aligned}$$

This equation shows that thermoinertial effects tends to reduce the effective inertial quantity of matter distribution of cylindrical stellar object with metric extra curvature terms.

6 STATIC CHARGED ANISOTROPIC/ISOTROPIC CYLINDERS

Here, we investigate the role of extended versions of structure scalars in the modeling of static non-rotating cylindrical systems in the presence of electromagnetic field and metric $f(R)$ corrections. In this section, we shall write the notation χ_i s for functions/constants arising after integration. The metric $f(R)$ field equations for static cylindrically symmetric spacetime modeled with anisotropic matter distribution are

$$\alpha_2 - \frac{B'^2}{B^2} - \frac{C''}{C} - \frac{B''}{B} = \frac{\kappa}{f_R} \left[\mu A^2 + \frac{2\pi A^2}{B^2} E^2 + \frac{A^2}{\kappa} \right. \\ \left. \times \left\{ \frac{f_R''}{A^2} - \frac{A'}{A} \frac{f_R'}{A^2} + \gamma_2 \right\} \right], \quad (65)$$

$$\frac{B'^2}{B^2} + \alpha_2 = \frac{\kappa}{f_R} \left[P_r A^2 - \frac{2\pi A^2}{B^2} E^2 + \frac{A^2}{\kappa} \left\{ -\gamma_2 - \frac{A'}{A} \frac{f_R'}{A^2} \right\} \right], \quad (66)$$

$$\frac{C''}{C} \frac{B^2}{A^2} + \beta_2 B^2 = \frac{\kappa}{f_R} \left[P_z B^2 + 2\pi E^2 + \frac{B^2}{\kappa} \left\{ \delta_2 - \frac{C'}{C} \frac{f_R'}{A^2} \right\} \right], \quad (67)$$

$$\frac{B''}{B} \frac{C^2}{A^2} + \beta_2 C^2 = \frac{\kappa}{f_R} \left[P_\phi C^2 + \frac{2\pi C^2}{B^2} E^2 + \frac{C^2}{\kappa} \left\{ \delta_2 - \frac{B'}{B} \frac{f_R'}{A^2} \right\} \right], \quad (68)$$

where

$$\begin{aligned} \alpha_2 = & \frac{A'}{A} \left(\frac{B'}{B} + \frac{C'}{C} \right), \quad \gamma_2 = \frac{R f_R - f}{2} + \left(\frac{B'}{B} + \frac{C'}{C} \right) \frac{f_R'}{A^2}, \\ \beta_2 = & \left(\frac{A''}{A} - \frac{A'^2}{A^2} \right) \frac{1}{A^2}, \quad \delta_2 = \frac{f - R f_R}{2} - \frac{f_R''}{A^2}. \end{aligned}$$

Now we would take some auxiliary terms, $\psi_1 \equiv \frac{A'}{A}$, $\psi_2 \equiv \frac{B'}{B}$, $\psi_3 \equiv \frac{C'}{C}$, $\psi_4 \equiv \frac{f'_R}{f_R}$. Under this formulation, Eqs.(66)-(69) with viable $f(R)$ model (42) become

$$\begin{aligned} & \frac{A^2 \kappa (1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \left(\mu - \frac{n\gamma R^{2n}}{2} \right. \\ & \times \frac{(\alpha R^n - \beta)}{2(1 + \gamma R^n)^2} + \frac{\alpha R^{2n}(1 - 2n) - \beta R^n(1 - n)}{2(1 + \gamma R^n)} + \frac{\phi_{tt}}{A^2} \\ & \left. + \frac{2\pi E^2}{B^2} \right) = -\psi'_2 - \psi_2^2 - \psi'_3 - \psi_3^2 + \psi_1\psi_2 + \psi_1\psi_3 \\ & - \psi_2\psi_3, \end{aligned} \quad (69)$$

$$\begin{aligned} & \frac{A^2 \kappa (1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} (P_r \\ & + \frac{n\gamma R^{2n}(\alpha R^n - \beta)}{2(1 + \gamma R^n)^2} - \frac{\alpha R^{2n}(1 - 2n) - \beta R^n(1 - n)}{2(1 + \gamma R^n)} \\ & + \frac{\phi_{rr}}{A^2} - \frac{2\pi E^2}{B^2}) = \psi_1\psi_2 + \psi_1\psi_3 + \psi_2\psi_3, \end{aligned} \quad (70)$$

$$\begin{aligned} & \frac{A^2 \kappa (1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} (P_z \\ & + \frac{n\gamma R^{2n}(\alpha R^n - \beta)}{2(1 + \gamma R^n)^2} - \frac{\alpha R^{2n}(1 - 2n) - \beta R^n(1 - n)}{2(1 + \gamma R^n)} \\ & + \frac{\phi_{zz}}{A^2} + \frac{2\pi E^2}{B^2}) = \psi'_1 + \psi'_3 + \psi_3^2, \end{aligned} \quad (71)$$

$$\begin{aligned} & \frac{A^2 \kappa (1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} (P_\phi \\ & + \frac{n\gamma R^{2n}(\alpha R^n - \beta)}{2(1 + \gamma R^n)^2} - \frac{\alpha R^{2n}(1 - 2n) - \beta R^n(1 - n)}{2(1 + \gamma R^n)} \\ & + \frac{\phi_{zz}}{A^2} + \frac{2\pi E^2}{B^2}) = \psi'_1 + \psi'_2 + \psi_2^2, \end{aligned} \quad (72)$$

where $\phi_{\alpha\beta}$ are dark source terms corresponding to static cylindrical system and can be evaluated from the corresponding values of $\psi_{\alpha\beta}$. It can be observed that all $\psi_{\alpha\beta}$ reduce to their respective $\phi_{\alpha\beta}$ under static environment of the metric variables. The couple of scalar variables corresponding to electric part of the Weyl tensor under auxiliary variables turn out to be

$$\mathbb{E}_S = \frac{1}{2A^2} (-\psi'_1 + \psi'_3 + \psi_3^2 + \psi_1\psi_2 - \psi_2\psi_3 - \psi_1\psi_3), \quad (73)$$

$$\mathbb{E}_K = \frac{1}{2A^2} (-\psi'_1 + \psi'_2 + \psi_2^2 - \psi_1\psi_2 + \psi_1\psi_3 - \psi_2\psi_3). \quad (74)$$

The structure scalar Y_T can be reshaped by means of auxiliary variables as

$$\psi'_1 + \psi_1\psi_2 + \psi_1\psi_3 = Y_T A^2. \quad (75)$$

Equations (69)-(72) give

$$\begin{aligned} & \frac{\kappa A^2 (1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \left(P_\phi + \frac{4\pi E^2}{B^2} \right. \\ & \left. - P_r + \frac{\phi_{\phi\phi}}{C^2} - \frac{\phi_{rr}}{A^2} \right) = \psi'_1 + \psi'_2 + \psi_2^2 - \psi_1\psi_2 - \psi_1\psi_3 - \psi_2\psi_3, \end{aligned} \quad (76)$$

$$\begin{aligned} & \frac{\kappa A^2 (1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \left(P_z + \frac{4\pi E^2}{B^2} \right. \\ & \left. - P_r + \frac{\phi_{zz}}{B^2} - \frac{\phi_{rr}}{A^2} \right) = \psi'_1 + \psi'_3 + \psi_3^2 - \psi_1\psi_2 - \psi_1\psi_3 - \psi_2\psi_3, \end{aligned} \quad (77)$$

$$\begin{aligned} & \frac{\kappa A^2 (1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} (P_\phi - P_z \\ & + \frac{\phi_{\phi\phi}}{C^2} - \frac{\mathcal{T}_{22}^{(D)}}{B^2}) = \psi'_2 + \psi_2^2 - \psi'_3 + \psi_3^2. \end{aligned} \quad (78)$$

Using Eqs.(21), (22), (73), (74), (76) and (77), we obtain

$$\psi'_1 - \psi_1\psi_2 = -Y_S A^2, \quad \psi'_1 + \psi_1\psi_3 = -Y_K A^2, \quad (79)$$

whose integrations, respectively, give

$$A = \chi_1 e^{\int B \left(\int \frac{-Y_S A^2}{B} dr \right) dr}, \quad A = \chi_2 e^{\int C \left(\int \frac{-Y_K A^2}{C} dr \right) dr},$$

which gives $B = B(A)$ and $C = C(A)$ or $\psi_1 = \psi_1(\psi_2)$, $\forall Y_S$ and $\psi_1 = \psi_1(\psi_3)$, $\forall Y_K$. This asserts that any auxiliary variable can be written in terms of the other one. For instance, RHS of Eq.(78) can be reexpressed containing the derivatives of ψ_1 . Once ψ_1 is calculated (with specific given value of $f(R)$ model) then differences of pressure gradients can easily be evaluated by using Eqs.(76) and (77). One can reduce the complexity of this differential equation by taking current value of Ricci invariant in $f(R)$ model. Further, Eqs.(22), (24) and (42) provide

$$\begin{aligned} & \frac{\kappa (1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} (P_\phi - P_r \\ & + \frac{4\pi E^2}{B^2} + \frac{\phi_{\phi\phi}}{C^2} - \frac{\phi_{rr}}{A^2}) = -(X_K + Y_K), \end{aligned} \quad (80)$$

while Eqs.(21), (23) and (42) give

$$\begin{aligned} & \frac{\kappa (1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} (P_z - P_r \\ & + \frac{4\pi E^2}{B^2} + \frac{\phi_{zz}}{B^2} - \frac{\phi_{rr}}{A^2}) = -(X_S + Y_S). \end{aligned} \quad (81)$$

Thus, it is concluded that any static cylindrical stellar solution coupled with anisotropic standard and metric $f(R)$ effective fluid distributions can be obtained from the set of any of two structure scalar triplets, i.e., (Y_K, Y_S, X_S) or (Y_K, Y_S, X_K) .

Next, we reduce the complexity of the system and consider static cylindrical celestial object filled with ideal matter configurations along with $f(R)$ corrections. Then, field equations (65)-(67) provide

$$\begin{aligned} & \alpha_2 - \frac{B'^2}{B^2} - \frac{C''}{C} - \frac{B''}{B} = \frac{\kappa}{f_R} \left[\mu A^2 + \frac{2\pi A^2}{B^2} E^2 + \frac{A^2}{\kappa} \right. \\ & \left. \times \left\{ \frac{f''_R}{A^2} - \frac{A'}{A} \frac{f'_R}{A^2} + \gamma_2 \right\} \right], \end{aligned} \quad (82)$$

$$\frac{B'^2}{B^2} + \alpha_2 = \frac{\kappa}{f_R} \left[P A^2 - \frac{2\pi A^2}{B^2} E^2 + \frac{A^2}{\kappa} \left\{ -\gamma_2 - \frac{A'}{A} \frac{f'_R}{A^2} \right\} \right], \quad (83)$$

$$\frac{C''}{C} \frac{B^2}{A^2} + \beta_2 B^2 = \frac{\kappa}{f_R} \left[P B^2 + 2\pi E^2 + \frac{B^2}{\kappa} \left\{ \delta_2 - \frac{C'}{C} \frac{f'_R}{A^2} \right\} \right], \quad (84)$$

$$\frac{B''}{B} \frac{C^2}{A^2} + \beta_2 C^2 = \frac{\kappa}{f_R} \left[P C^2 + \frac{2\pi C^2}{B^2} E^2 + \frac{C^2}{\kappa} \left\{ \delta_2 - \frac{B'}{B} \frac{f'_R}{A^2} \right\} \right]. \quad (85)$$

Substituting the value of P from Eq.(84) in Eq.(85), we get

$$\frac{C''}{C} + \frac{C'}{C} \frac{f'_R}{f_R} = \frac{B''}{B} + \frac{B'}{B} \frac{f'_R}{f_R}, \quad (86)$$

which by means of auxiliary functions can be recasted as

$$\psi'_3 + \psi_3^2 + \psi_3\psi_4 = \psi'_2 + \psi_2^2 + \psi_2\psi_4. \quad (87)$$

This is the Ricatti equation whose generic solution can be written as

$$\psi_3 = \psi_2 + \frac{1}{k(r)}, \quad (88)$$

where

$$k(r) = e^{\int (2\psi_2 + \psi_4) dr} \left[\int e^{\int -(2\psi_2 + \psi_4) dr} dr + \chi_3 \right]. \quad (89)$$

Using k from above equation in Eq.(88) and after some manipulation, we obtain

$$C = B\gamma \exp \left[B^2 f_R (\chi_3 + \int \frac{B^2(1 + \gamma R^n)^2}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} dr \right], \quad (90)$$

in which one should need to consider regularity conditions, i.e., $C(t, 0) = 0$, for feasible model evaluation.

7 CONCLUSION

The $f(R)$ gravity models, in which the Lagrangian for the gravitation is a function of curvature scalar, has attracted the attention of many researchers. However, to find the solutions of such field equations is still rare. In this manuscript, we have presented a procedure indicating how one can write the field equations using some set of scalar function in static form. The philosophy of this work is to demonstrate dynamical variables for cylindrical collapsing object and to present a procedure to find the solutions of field equations with the properties of modified gravity having positive and negative powers of curvature. We consider that the cylindrical body is filled with anisotropic matter in the presence of dissipation with diffusion approximation under the influence of electromagnetic field. The modified field equations and conservation laws are explored for the systematic construction of our work. We explore the expressions for the shear tensor as well as the Weyl tensor and analyze some crucial aspects as:

- There are two scalars associated with the shear tensor unlike the spherical case where it has only one such scalar. For the shearfree system these two scalars must disappear.
- The Weyl tensor can be decomposed into two constituent tensors namely, its electric and magnetic parts. It is observed that magnetic part vanishes in the spherical fluid configuration due to symmetry of the problem, but in cylindrical star formalism, the magnetic part is non-zero implying a crucial role in our dynamical analysis.

Next, we have disintegrated the Riemann tensor using comoving vectors which are orthogonal to each other. We found three tensors called the electric, magnetic and dual of the Riemann tensor. Further, these tensors are written in terms of its trace and trace-free scalar parts called the structure scalars. Using the modified field equations, we have obtained these scalar parts in terms of material variables like anisotropic stresses, energy density, heat flux and particularly electric charge and $f(R)$ higher curvature terms. It is found that there are three scalar functions associated with the electric and dual part of the Riemann tensor while two

scalars are associated with its magnetic part. We would like to stress here that magnetic part of the Riemann tensor is due to the presence of magnetic part of Weyl tensor as well as presence of heat radiations and if the magnetic part of the Weyl tensor vanishes in dissipation less case then it leads to the vanishing of Riemann tensor's magnetic part. It is interesting to note that in the spherical case both the magnetic parts of the Weyl as well as the Riemann tensors are zero for non-dissipative systems. A particular viable $f(R)$ model is used to examine the role of dark source terms on the evolution of these structure scalars. We have also explored two evolution equations using these structure scalars to investigate their role on the homogeneity of cylindrical object.

We have obtained an expression for the mass function in $f(R)$ gravity model for cylindrical object in comparison with the Misner-Sharp mass in spherical system. We have also explained this mass using the modified structure scalars obtained in the previous section. After that we found mass radius relationship plots for different parameter values. We found that one can get even more compact and massive cylindrical stellar systems (as compared to GR) for modified gravities induced by $f(R) = \alpha R^2$, $f(R) = \alpha R^2 - \beta R$ and $f(R) = \frac{\alpha R^2 - \beta R}{1 + \gamma R}$ models.

Since our system is dealing with the heat dissipation in diffusion approximation so we have explored the transportation of heat phenomena through second order thermodynamical theory defined by Müller and Israel. We have coupled the transport equation with the dynamical equation obtained from the second non-zero equation of the conservation law to investigate the collapsing behavior of the system as given in Eq.(64). We found α to be responsible for the decrease and increase in the gravitational mass of the system and indicate how thermal effects reduce the effective inertial mass. We observe three main possibilities on this value for the collapsing behavior as follows:

- If $\alpha \rightarrow 1$, then the inertial mass of the system approaches to zero value indicating null system inertial force thereby leading the system towards gravitational attraction and consequently causes the collapse.
- If $\alpha > 1$, then it makes increment in the inertial mass of the system. This behavior is consistent with the equivalence principle indicating the expanding behavior of the system.
- When $0 < \alpha < 1$, then inertial mass density keeps on moving in the decreasing state.

Since to find the solutions of the modified field equations is huge task and particularly with dissipation and electromagnetic effects so in the last section we have presented a framework to explore the solutions of $f(R)$ field equations in the static background.

It is worthy to stress that all GR already established results (Herrera, Di Prisco & Ospino 2012) can be obtained by taking $f(R) \rightarrow R$ and $s = 0$.

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APPENDIX A:

The $f(R)$ corrections in Eqs.(39) and (40) are

$$\begin{aligned}
 D_0 = & \frac{1}{\kappa} \left[\left\{ \left(\dot{f}_R \frac{A'}{A} + f'_R \frac{\dot{A}}{A} - \dot{f}'_R \right) \frac{1}{A^4} \right\}_{,1} + \frac{1}{A^2} \left\{ \frac{f'_R}{A^2} \right. \right. \\
 & \times \left(\frac{B'}{B} - \frac{A'}{A} + \frac{C'}{C} \right) - \left(\frac{\dot{C}}{C} + \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) \frac{f_R}{A^2} + \frac{f''_R}{A^2} \\
 & + \left(\frac{Rf_R - f}{2} \right) \left. \right\}_{,0} + \frac{\dot{A}}{A} \left\{ \frac{\ddot{f}_R}{A^2} + \frac{f''_R}{A^2} - 2 \frac{A' f'_R}{A^3} - 2 \frac{\dot{A} f_R}{A^3} \right\} \\
 & \times \frac{1}{A^2} + \frac{\dot{B}}{B} \left\{ \frac{\ddot{f}_R}{A^2} - \left(\frac{\dot{B}}{B} + \frac{\dot{A}}{A} \right) \frac{f_R}{A^2} + \frac{f'_R}{B^2} \left(\frac{B'}{B} - \frac{A'}{A} \right) \right\} \\
 & \frac{1}{A^2} + \left(\frac{\dot{A}}{A} f'_R - \dot{f}_R + \frac{A'}{A} \dot{f}_R \right) \left(\frac{4A'}{A} + \frac{C'}{C} + \frac{B'}{B} \right) \frac{1}{A^4} \\
 & + \frac{2}{A^2} \left\{ - \left(\frac{A'}{A} - \frac{C'}{C} \right) \frac{f'_R}{B^2} + \frac{\ddot{f}_R}{A^2} - \frac{f_R}{A^2} \left(\frac{\dot{C}}{C} + \frac{\dot{A}}{A} \right) \right\} \frac{\dot{C}}{C} \Bigg], \tag{A1}
 \end{aligned}$$

$$\begin{aligned}
 D_1 = & \frac{1}{\kappa} \left[\left\{ \frac{1}{A^4} \left(\frac{A'}{A} \dot{f}_R - \dot{f}_R + \frac{\dot{A}}{A} f'_R \right) \right\}_{,0} + \frac{1}{A^2} \left\{ \frac{\ddot{f}_R}{A^2} - \frac{f'_R}{B^2} \right. \right. \\
 & \times \left(\frac{B'}{B} + \frac{A'}{A} + \frac{C'}{C} \right) + \frac{f - Rf_R}{2} - \frac{f_R}{A^2} \left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A} + \frac{\dot{C}}{C} \right) \left. \right\}_{,1} \\
 & + \frac{1}{A^2} \left\{ \frac{f''_R}{A^2} - \left(\frac{A'}{A} + \frac{B'}{B} \right) \frac{f'_R}{B^2} + \left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A} \right) \frac{f_R}{A^2} \right\} \frac{B'}{B} \\
 & + \left\{ \frac{f''_R}{A^2} + \frac{\ddot{f}_R}{A^2} - 2 \frac{A' f'_R}{A^3} - 2 \frac{\dot{A} f_R}{A^3} \right\} \frac{A'}{A^3} + \frac{1}{A^2} \left\{ \frac{f''_R}{A^2} - \frac{f'_R}{A^2} \right. \\
 & \times \left(\frac{A'}{A} + \frac{C'}{C} \right) - \frac{f_R}{A^2} \left(\frac{\dot{C}}{C} - \frac{\dot{A}}{A} \right) \left. \right\} \frac{C'}{C} - \frac{1}{A^4} \left(\dot{f}_R - \frac{\dot{A}}{A} f'_R \right.
 \end{aligned}$$

$$- \frac{A'}{A} \dot{f}_R \left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C} + \frac{4\dot{A}}{A} \right) \Bigg]. \quad (\text{A2})$$

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In view of $f(R)$ model, the modified versions of structure scalars are

$$Y_T^{(G)} = \frac{\kappa(1 + \gamma R^n)^2 (\mu + P_z + P_r + P_\phi)}{2nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]}, \quad (\text{A3})$$

$$Y_T^{(C)} = \frac{2\kappa(1 + \gamma R^n)^2 \pi E^2}{nR^{n-1}B^2[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} = Y_S^{(C)} = -Y_K^{(C)}, \quad (\text{A4})$$

$$Y_T^{(D)} = \frac{(1 + \gamma R^n)^2}{2nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \left[\frac{\alpha R^{2n}(1 - 2n)}{2(1 + \gamma R^n)} - \frac{\beta R^n(1 - n)}{2(1 + \gamma R^n)} + \frac{n\gamma R^{2n}(\alpha R^n - \beta)}{2(1 + \gamma R^n)^2} + \frac{1}{A^2} (\psi_{tt} + \psi_{rr}) + \frac{\psi_{zz}}{B^2} + \frac{\psi_{\phi\phi}}{C^2} \right], \quad (\text{A5})$$

$$Y_S^{(G)} = \mathbb{E}_S - \frac{\kappa(1 + \gamma R^n)^2 (P_z - P_r)}{2nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]}, \quad (\text{A6})$$

$$Y_S^{(D)} = \frac{\kappa(1 + \gamma R^n)^2}{2nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \left(\frac{\psi_{rr}}{A^2} - \frac{\psi_{zz}}{B^2} \right), \quad (\text{A7})$$

$$Y_K^{(G)} = \mathbb{E}_K - \frac{\kappa(1 + \gamma R^n)^2 (P_\phi - P_r)}{2nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \quad (\text{A8})$$

$$Y_K^{(D)} = \frac{\kappa(1 + \gamma R^n)^2}{2nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \left(\frac{\psi_{rr}}{A^2} - \frac{\psi_{zz}}{C^2} \right), \quad (\text{A9})$$

$$X_S^{(G)} = -\mathbb{E}_S - \frac{\kappa(1 + \gamma R^n)^2 (P_z - P_r)}{2nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]}, \quad (\text{A10})$$

$$X_S^{(C)} = -\frac{2\pi(1 + \gamma R^n)^2 \kappa E^2}{nB^2 R^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} = X_K^{(C)} = -X_T^{(C)}, \quad (\text{A11})$$

$$X_S^{(D)} = \frac{\kappa(1 + \gamma R^n)^2}{2nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \left(\frac{\psi_{rr}}{A^2} - \frac{\psi_{zz}}{B^2} \right), \quad (\text{A12})$$

$$X_K^{(G)} = -\mathbb{E}_K - \frac{\kappa(1 + \gamma R^n)^2 (P_\phi - P_r)}{2nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \quad (\text{A13})$$

$$X_K^{(D)} = -\frac{\kappa(1 + \gamma R^n)^2}{2nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]} \left(\frac{\psi_{\phi\phi}}{C^2} - \frac{\psi_{rr}}{A^2} \right), \quad (\text{A14})$$

$$X_T^{(G)} = \frac{\kappa(1 + \gamma R^n)^2 (\mu)}{nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]}, \quad (\text{A15})$$

$$X_T^{(D)} = \frac{\kappa(1 + \gamma R^n)^2 \psi_{tt}}{2nA^2 R^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]}, \quad (\text{A16})$$

$$Z_q^{(G)} = \frac{\kappa(1 + \gamma R^n)^2 q}{2nR^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]}, \quad Z_q^{(C)} = 0, \quad (\text{A17})$$

$$Z_q^{(D)} = -\frac{\kappa(1 + \gamma R^n)^2 \psi_{tr}}{2nA^2 R^{n-1}[(1 + \gamma R^n)(2\alpha R^n - \beta) - \gamma R^n(\alpha R^n - \beta)]}, \quad Z_H = 2H. \quad (\text{A18})$$